On Roots of the Macdonald Function

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07 May 2014

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Introduction

Setting

- Outgoing solutions to the “radial wave equation"
- Transparent radiation boundary conditions on spherical domain
- Dirichlet-to-Neumann map, Macdonald function

Outline

- Radial wave equation and the Macdonald function
- Non-reflecting boundary conditions
- Identities for roots of the Macdonald function
- Other work

Radial Wave Equation

Overview

- Ordinary wave equation on $\mathbb{R}^3$, ($c = 1$ here): $\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$
- Laplace operator in spherical coordinates:

$$\Delta\psi = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\psi + \frac{1}{r^2}\Delta_{S^2}\psi, \qquad \Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

- Generic solution in terms of $Y_{\ell m}(\theta,\phi)$ of Δ_{S^2}, where

$$\Delta_{S^2}Y_{\ell m}(\theta,\phi) = -\ell(\ell+1)Y_{\ell m}(\theta,\phi)$$

is:

$$\psi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\psi_{\ell m}(t,r)Y_{\ell m}(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\Psi_{\ell m}(t,r)}{r}Y_{\ell m}(\theta,\phi).$$

Overview

- "multipole" solution $\psi(t, r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$:

$$\psi_{\ell m}(t,r)Y_{\ell m}(\theta,\phi) = r^{-1}\Psi_{\ell m}(t,r)Y_{\ell m}(\theta,\phi)$$

- Eliminating angular dependence, multipole coefficients $\psi_{\ell m}$ and $\Psi_{\ell m}$ satisfy:

$$-\partial_t^2\psi_{\ell m} + \partial_r^2\psi_{\ell m} + \frac{2}{r}\partial_r\psi_{\ell m} - \frac{\ell(\ell+1)}{r^2}\psi_{\ell m} = 0,$$

$$-\partial_t^2\Psi_{\ell m} + \partial_r^2\Psi_{\ell m} - \frac{\ell(\ell+1)}{r^2}\Psi_{\ell m} = 0.$$

Radial Wave Equation

Overview

- The Macdonald function $K_\nu(z)$ is a solution to the modified Bessel equation:

$$z^2 w'' + zw' - (z^2 + \nu^2)w = 0$$

- For half-integer order,

$$K_{\ell+1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} W_\ell(z), \qquad W_\ell(z) = \sum_{k=0}^{\ell} \frac{c_{\ell k}}{z^k}, \qquad c_{\ell k} = \frac{1}{2^k k!} \frac{(\ell+k)!}{(\ell-k)!}$$

- $K_{\ell+1/2}(z)$ can also be expressed using the monic *Bessel polynomial* $p_\ell(z)$:

$$K_{\ell+1/2}(z) = \sqrt{\frac{\pi}{2z}} \frac{e^{-z}}{z^\ell} p_\ell(z), \qquad p_\ell(z) = \sum_{k=0}^{\ell} c_{\ell k} z^{\ell-k}$$

Radial Wave Equation

Overview

- The set $\{b_{\ell j}/(\ell + 1/2) : j = 1, \ldots, \ell\}$ is the collection of roots scaled by the Bessel order $\nu = \ell + 1/2$.
- These roots accumulate on a fixed transcendental curve in the left-half plane, a parametrization of which is given by:

$$z(\lambda) = -\sqrt{\lambda^2 - \lambda \tanh \lambda} \pm \mathrm{i}\sqrt{\lambda \coth \lambda - \lambda^2}$$

for $\lambda \in [0, \lambda_0]$, where $\lambda_0 \simeq 1.19967864025773$ solves $\tanh \lambda_0 = 1/\lambda_0$.

Radial Wave Equation

Scaled zeros $\dfrac{b_{\ell j}}{(\ell+1/2)}$ of $K_{\ell+1/2}(z)$ and $W_\ell(z)$:

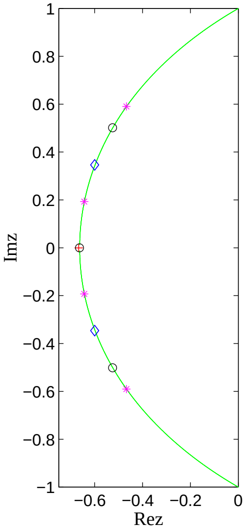
$+$ $K_{3/2}(z)=\sqrt{\dfrac{\pi}{2z}}e^{-z}\left(1+\dfrac{1}{z}\right)$

$\diamond$ $K_{5/2}(z)=\sqrt{\dfrac{\pi}{2z}}e^{-z}\left(1+\dfrac{3}{z}+\dfrac{3}{z^2}\right)$

$\circ$ $K_{7/2}(z)=\sqrt{\dfrac{\pi}{2z}}e^{-z}\left(1+\dfrac{6}{z}+\dfrac{15}{z^2}+\dfrac{15}{z^3}\right)$

$*$

$K_{9/2}(z)=\sqrt{\dfrac{\pi}{2z}}e^{-z}\left(1+\dfrac{10}{z}+\dfrac{45}{z^2}+\dfrac{105}{z^3}+\dfrac{105}{z^4}\right)$



Radial Wave Equation

- Radial Wave Equation:

$$-\partial_t^2 \Psi_\ell + \partial_r^2 \Psi_\ell - \frac{\ell(\ell+1)}{r^2}\Psi_\ell = 0$$

- For $\ell = 0$, generic solutions from factoring wave equation:

$$\underrightarrow{f(t-r)}, \qquad \underleftarrow{g(t+r)}$$

Lemma

For $\ell > 0$, generic outgoing solution:

$$\Psi_\ell(t,r) = \sum_{k=0}^{\ell} \frac{c_{\ell k}}{r^k} f^{(\ell-k)}(t-r) \qquad c_{\ell k} = \frac{1}{2^k k!}\frac{(\ell+k)!}{(\ell-k)!}$$

Radial Wave Equation

By induction,

$$\Psi_\ell = -\Big(\frac{\partial}{\partial r} - \frac{\ell}{r}\Big)\Psi_{\ell-1}, \qquad D_\ell^+ = -\Big(\frac{\partial}{\partial r} - \frac{\ell}{r}\Big)$$

If

$$-\partial_t^2\Psi_{\ell-1} + \partial_r^2\Psi_{\ell-1} - \frac{\ell(\ell-1)}{r^2}\Psi_{\ell-1} = 0,$$

Then

$$-\partial_t^2(D_\ell^+\Psi_{\ell-1}) + \partial_r^2(D_\ell^+\Psi_{\ell-1}) - \frac{\ell(\ell+1)}{r^2}D_\ell^+\Psi_{\ell-1} = 0.$$

Non-reflecting Boundary Conditions

Lemma [needed to compute LT of $\Psi_\ell(t,r)$]

Figure: D and open region $r > r_B - \delta$

Assume $f(u)$ supported on $[-r_B + \delta, -r_0 - \delta] = D$, $f \in C_0^\infty(D)$.

For $r > r_B - \delta$, Fourier-Laplace transform of $f^{(\ell-k)}(t-r)$ is

$$s^{\ell-k} e^{-sr} a(s), \qquad a(s) = \int_{-r_B+\delta}^{-r_0-\delta} e^{-su} f(u) du.$$

proof

$$\int_0^\infty e^{-st} f^{(\ell-k)}(t-r) dt = e^{-sr} \int_{-r}^\infty e^{-su} f^{(\ell-k)}(u) du$$

$$= e^{-sr} \int_{-r_B+\delta}^{-r_0-\delta} e^{-su} f^{(\ell-k)}(u) du, \qquad -r < -r_B + \delta$$

Laplace Transform of $\Psi_\ell(t, r)$

Therefore, for $r = r_B$,

$$\widehat{\Psi}_\ell(s, r) = \sum_{k=0}^{\ell} \frac{c_{\ell k}}{r^k} \left(s^{\ell-k} e^{-sr} a(s)\right)$$

$$= a(s) s^\ell e^{-sr} W_\ell(sr), \qquad W_\ell(sr) = \sum_{k=0}^{\ell} \frac{c_{\ell k}}{z^k}$$

$$s\widehat{\Psi}_\ell(s, r) + \partial_r \widehat{\Psi}_\ell(s, r) = \frac{1}{r}(sr)\frac{W_\ell'(s, r)}{W_\ell(s, r)}\widehat{\Psi}_\ell = \frac{1}{r}\sum_{j=1}^{\ell} \frac{b_{\ell j}/r}{s - b_{\ell j}/r}\widehat{\Psi}_\ell,$$

Laplace Convolution Theorem:

$$\partial_t \Psi_\ell(t, r) + \partial_r \Psi_\ell(t, r) = \frac{1}{r}\int_0^t \Omega_\ell(t - t', r)\Psi_\ell(t', r)dt', \qquad \Omega_\ell(t, r) = \sum_{k=1}^{\ell} \frac{b_{\ell k}}{r} e^{\frac{b_{\ell k}}{r} t}$$

Identities for Roots of Macdonald function

Sketch of Main Results

- Substitute $\Psi_\ell(t,r) = \sum_{k=0}^{\ell} \frac{1}{r^k} c_{\ell k} f^{(\ell-k)}(t-r)$ into

$$\partial_t \Psi_\ell(t,r) + \partial_r \Psi_\ell(t,r) = \sum_{n=1}^{\ell} \frac{b_{\ell n}}{r^2} \int_0^t e^{\frac{b_{\ell n}}{r}(t-t')} \Psi_\ell(t',r) dt'$$

- Result:

$$-\sum_{k=1}^{\ell} \frac{k}{r^{k+1}} c_{\ell k} f^{(\ell-k)}(t-r) = \sum_{n=1}^{\ell} \frac{b_{\ell n}}{r^2} \sum_{k=0}^{\ell} \frac{1}{r^k} c_{\ell k} I^{(\ell-k)}[b_{\ell n}, r, f]$$

- Here $I^{(p)}[b,r,f] \equiv \int_0^t e^{\frac{b}{r}(t-t')} f^{(p)}(t'-r) dt'$
- Work into form (argument sketched later...integration by parts)

$$0 = \sum_{k=1}^{\ell} r^{-(k+1)} E_k f^{(\ell-k)}(u), \quad E_k = k c_{\ell k} + \sum_{n=1}^{\ell} b_{\ell n} \sum_{q=1}^{k} c_{\ell,q-1} (b_{\ell n})^{k-q}$$

Sketch of Main Results

- Starting with (last line of last slide)

$$0 = \sum_{k=1}^{\ell} r^{-(k+1)} E_k f^{(\ell-k)}(u), \quad E_k = kc_{\ell k} + \sum_{n=1}^{\ell} b_{\ell n} \sum_{q=1}^{k} c_{\ell,q-1}(b_{\ell n})^{k-q},$$

  isolate terms $E_k f^{(\ell-k)}(u)$ with operator $Q = (\partial_t + \partial_r)r^2$.
- Example ($\ell = 3$):

$$Q\Big[\frac{1}{r^2}E_1 f''(u) + \frac{1}{r^3}E_2 f'(u) + \frac{1}{r^4}E_3 f(u)\Big] = -\frac{1}{r^2}E_2 f'(u) - \frac{2}{r^3}E_3 f(u)$$

$$Q^2\Big[\frac{1}{r^2}E_1 f''(u) + \frac{1}{r^3}E_2 f'(u) + \frac{1}{r^4}E_3 f(u)\Big] = \frac{6}{r^2}E_3 f(u)$$

- Profile f arbitrary $\implies E_3 = 0 \implies E_2 = 0 \implies E_1 = 0$.

Technical Lemma (History Integrals)

Lemma

Let $r = r_B$ and $f \in C_0^\infty(D)$, and define

$$I^{(p)}[b, r, f] = \int_0^t e^{(b(t-t')/r)} f^{(p)}(t' - r) dt'$$

Then, we have

$$I^{(p)}[b, r, f] = (b/r)^p I^{(0)}[b, r, f] + \sum_{j=1}^{p} (b/r)^{p-j} f^{(j-1)}(t - r)$$

Technical Lemma (History Integrals)

Proof by induction

We show:

$$I^{(p)}[b,r,f] = (b/r)^p I^{(0)}[b,r,f] + \sum_{j=1}^{p} (b/r)^{p-j} \left[f^{(j-1)}(t-r) - e^{(bt/r)} f^{(j-1)}(-r) \right] \qquad (**)$$

The second term within the square brackets vanishes since $r = r_B \notin D$. Note there are no boundary terms at $t' = 0$, only $t' = t$. Integration by parts establishes the last formula for $p = 1$. Similarly,

$$I^{(p)}[b,r,f] = (b/r) I^{(p-1)}[b,r,f] + f^{(p-1)}(t-r) - e^{(bt/r)} f^{(p-1)}(-r).$$

Assuming now that $(**)$ holds with p replaced by $p-1$, we insert the $p-1$ result into the last equation, thereby recovering $(**)$ and verifying the induction step.

Main Result

Theorem

The roots $b_{\ell j} : j = 1, ..., \ell$ of the ℓ^{th} degree polynomial

$$p_\ell(z) = \sum_{k=0}^{\ell} c_{\ell k} z^{\ell-k}$$

also obey the following set of ℓ algebraic equations: (Newton's identities!)

$$-kc_{\ell k} = \sum_{n=1}^{\ell} b_{\ell n} \sum_{q=1}^{k} c_{\ell,q-1} (b_{\ell n})^{k-q}, \qquad k = 1, ..., \ell$$

Here we assume $\ell \geq 1$.

Main Result: careful proof (1)

proof

Taking $r = r_B$ and $t > 0$, we get

$$\frac{1}{r}\int_0^t \Omega_\ell(t-t',r)\Psi_\ell(t',r)dt' = \sum_{n=1}^{\ell}(b_{\ell n}/r^2)\int_0^t \exp(b_{\ell n}(t-t')/r)\Psi_\ell(t',r)dt'$$

$$= \sum_{n=1}^{\ell}(b_{\ell n}/r^2)\sum_{k=0}^{\ell} r^{-k} c_{\ell k} I^{(\ell-k)}[b_{\ell n}, r, f].$$

Previous lemma gives

$$I^{(\ell-k)}[b_{\ell n}, r, f] = (b_{\ell n}/r)^{\ell-k} I^{(0)}[b_{\ell n}, r, f] + \sum_{j=1}^{\ell-k}(b_{\ell n}/r)^{\ell-k-j} f^{(j-1)}(t-r).$$

Main Result: careful proof continued...(2)

Combination of the last two equations yields

$$\frac{1}{r}\int_0^t \Omega_\ell(t-t',r)\Psi_\ell(t',r)dt' = r^{-(\ell+2)}\sum_{n=1}^{\ell} b_{\ell n} I^{(0)}[b_{\ell n},r,f]\sum_{k=0}^{\ell} c_{\ell k}(b_{\ell n})^{\ell-k}$$

$$+\sum_{n=1}^{\ell}(b_{\ell n}/r^2)\sum_{k=0}^{\ell} r^{-k}c_{\ell k}\sum_{j=1}^{\ell-k}(b_{\ell n}/r)^{\ell-k-j}f^{(j-1)}(t-r).$$

Main Result: careful proof continued...(3)

Since $\sum_{k=0}^{\ell} c_{\ell k}(b_{\ell n})^{\ell-k} = p_\ell(b_{\ell n})$ is the Bessel polynomial evaluated at one of its roots, the first term on the right-hand side of the last expression vanishes. Whence, up to now

$$\frac{1}{r}\int_0^t \Omega_\ell(t-t',r)\Psi_\ell(t',r)dt' = \sum_{n=1}^{\ell}(b_{\ell n}/r^2)\sum_{k=0}^{\ell} r^{-k}c_{\ell k}\sum_{j=1}^{\ell-k}(b_{\ell n}/r)^{\ell-k-j}f^{(j-1)}(t-r).$$

Main Result: careful proof continued...(4)

Within the sum over k, the sum over j is empty when $k = \ell$. Therefore, here we may replace $\sum_{k=0}^{\ell}$ by $\sum_{k=0}^{\ell-1}$. Re-indexing $q = k + 1$, yields

$$\frac{1}{r}\int_0^t \Omega_\ell(t-t',r)\Psi_\ell(t',r)dt'$$

$$= \sum_{n=1}^{\ell}(b_{\ell n}/r^2)\sum_{q=1}^{\ell} r^{-(q-1)}c_{\ell,q-1}\sum_{j=1}^{\ell-q+1}(b_{\ell n}/r)^{\ell-q-j+1}f^{(j-1)}(t-r).$$

Main Result: careful proof continued...(5)

The double sum $\sum_{q=1}^{\ell}\sum_{j=1}^{\ell-q+1}(terms)_{jq}$ is equivalent to $\sum_{j=1}^{\ell}\sum_{q=1}^{\ell-j+1}(terms)_{jq}$. We may group the two inner sums and make this exchange, thereby reaching

$$\frac{1}{r}\int_0^t \Omega_\ell(t-t',r)\Psi_\ell(t',r)dt' =$$

$$\sum_{n=1}^{\ell} b_{\ell n}\sum_{j=1}^{\ell} r^{j-(\ell+2)}\sum_{q=1}^{\ell-j+1} c_{\ell,q-1}(b_{\ell n})^{\ell-q-j+1} f^{(j-1)}(t-r).$$

Through re-indexing of sums we get to

$$\frac{1}{r}\int_0^t \Omega_\ell(t-t',r)\Psi_\ell(t',r)dt' = \sum_{n=1}^{\ell} b_{\ell n} \sum_{k=1}^{\ell} r^{-(k+1)} \sum_{q=1}^{k} c_{\ell,q-1}(b_{\ell n})^{k-q} f^{(\ell-k)}(t-r),$$

which is the desired form. Recalling that we have set $r = r_B$,

$$-\sum_{k=1}^{\ell} kr^{-(k+1)} c_{\ell k} f^{(\ell-k)}(t-r) = \sum_{k=1}^{\ell} r^{-(k+1)} \sum_{n=1}^{\ell} b_{\ell n} \sum_{q=1}^{k} c_{\ell,q-1}(b_{\ell n})^{k-q} f^{(\ell-k)}(t-r),$$

and we may express that last equation as

$$0 = \sum_{k=1}^{\ell} r^{-(k+1)} E_k f^{(\ell-k)}(u), \qquad E_k = kc_{\ell k} + \sum_{n=1}^{\ell} b_{\ell n} \sum_{q=1}^{k} c_{\ell,q-1}(b_{\ell n})^{k-q}.$$

Here $u = t - r$ is retarded time.

Also in thesis

- Numerical check using **Mathematica**
- Watson's argument showing poles are simple and in left-half plane.

real and imaginary parts of roots for $\ell = 20$

-1.3567424283153313290440 9805E+001	8.6773625495579827756390 1221E-001
-1.3412597143606601898430 2504E+001	2.6054001471794975424052 8038E+000
-1.3098822474577163249021 3023E+001	4.3498649117914624014826 5878E+000
-1.2617281316609851725011 7887E+001	6.1064798700523964612462 3649E+000
-1.1953090802499987253529 8649E+001	7.8820584342474498915743 8123E+000
-1.1082580333731152027208 2869E+001	9.6860932418285785111963 7990E+000
-9.9677624788603911246162 0511E+000	1.1533114728516246638641 2696E+001
-8.5438957268500319087321 3213E+000	1.3448045273419699724711 2442E+001
-6.6855268782951902009239 5369E+000	1.5481306187923618554333 0417E+001
-4.0710185618163173220852 3537E+000	1.7771869068885456122597 3949E+001

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